Exact Analysis of the Adiabatic Invariants in Time-Dependent Harmonic Oscillator

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(Dated: February 8, 2008)

The theory of adiabatic invariants has a long history and important applications in physics but is rarely rigorous. Here we treat exactly the general time-dependent 1-D harmonic oscillator, $\ddot{q} + \omega^2(t)q = 0$ which cannot be solved in general. We follow the time-evolution of an initial ensemble of phase points with sharply defined energy E_0 and calculate rigorously the distribution of energy E_1 after time T, and all its moments, especially its average value $\bar{E}_1$ and variance μ^2. Using our exact WKB-theory to all orders we get the exact result for the leading asymptotic behaviour of μ^2.

PACS numbers: 05.45.-a, 45.20.-d, 45.30.+s, 47.52.+j

Adiabatic invariants, usually denoted by I, in time dependent dynamical systems (not necessarily Hamiltonian), are approximately conserved during a slow process of changing system parameters over a long typical time scale T. This statement is asymptotic in the sense that the conservation is exact in the limit $T \to \infty$, whilst for finite T we see the deviation $\Delta I = I_f - I_i$ of final value of I_f from its initial value I_i and would like to calculate ΔI. Here we just remind that for the one-dimensional harmonic oscillator it is known since Ehrenfest that the adiabatic invariant for $T = \infty$ is $I = E/\omega$, which is the ratio of the total energy $E = E(t)$ and the frequency of the oscillator $\omega(t)$, both being a function of time. Of course, $2\pi I$ is exactly the area in the phase plane (q,p) enclosed by the energy contour of constant E. A general introductory account can be found in [1] and references therein, especially [2, 3]. However, in the literature this ΔI is not even precisely defined. As a consequence of that there is considerable confusion about its meaning. Let us just mention the case of periodic parametric resonance in one-dimensional harmonic oscillator where the driving is periodic and yet e.g. the total energy of the system can grow indefinitely for certain system parameter values. (In this work we give a precise meaning to these and similar statements.) Therefore to be on rigorous side we must carefully define what we mean by ΔI. This can be done by considering an ensemble of initial conditions at time $t = 0$ just before the adiabatic process starts. Of course, there is a vast freedom in choosing such ensembles. In an integrable conservative Hamiltonian system the most natural and the most important choice is taking as the initial ensemble all phase points uniformly distributed on the initial N-torus, uniform w.r.t. the angle variables. Such an ensemble has a sharply defined initial energy E_0. Then we let the system evolve in time, not necessarily slowly, and calculate the probability distribution $P(E_1)$ of the final energy E_1, or of other dynamical quantities.

This is in general a difficult problem, but in this work we confine ourself to the one-dimensional general time-dependent harmonic oscillator, so $N = 1$, described by the Newton equation

$$\ddot{q} + \omega^2(t)q = 0 \tag{1}$$

and work out rigorously $P(E_1)$. Given the general dependence of the oscillator's frequency $\omega(t)$ on time t the calculation of $q(t)$ is already a very difficult unsolvable problem. In the sense of mathematical physics (1) is exactly equivalent to the one-dimensional stationary Schrödinger equation: the coordinate q appears instead of the probability amplitude ψ, time t appears instead of the coordinate x and $\omega^2(t)$ plays the role of $E - V(x)$ = energy – potential. In this paper we solve the above stated problem for the general one-dimensional harmonic oscillator, but the details of our calculations are delegated to another publication [4].

We begin by defining the system by giving its Hamilton function $H = H(q,p,t)$, whose numerical value $E(t)$ at time t is precisely the total energy of the system at time t, and for the one-dimensional harmonic oscillator this is

$$H = \frac{p^2}{2M} + \frac{1}{2}M\omega^2(t)q^2, \tag{2}$$

where q, p, M, ω are the coordinate, the momentum, the mass and the frequency of the linear oscillator, respectively. The dynamics is linear in q, p, as described by (1), but nonlinear as a function of $\omega(t)$ and therefore is subject to the nonlinear dynamical analysis. By using the index 0 and 1 we denote the initial ($t = t_0$) and final ($t = t_1$) value of the variables, and by $T = t_1 - t_0$ we denote the time interval of changing the parametrs of the system.

We consider the phase flow map (we shall call it transition map)

$$\Phi : \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \mapsto \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}. \tag{3}$$

Because equations of motion are linear in q and p, and since the system is Hamiltonian, Φ is a linear area pre-

serving map, that is,

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4}$$

with $\det(\Phi) = ad - bc = 1$. Let $E_0 = H(q_0, p_0, t = t_0)$ be the initial energy and $E_1 = H(q_1, p_1, t = t_1)$ be the final energy, that is,

$$E_1 \quad = \quad \frac{1}{2}\left(\frac{(cq_0 + dp_0)^2}{M} + M\omega_1^2(aq_0 + bp_0)^2\right). \tag{5}$$

Introducing the new coordinates, namely the action $I = E/\omega$ and the angle ϕ,

$$q_0 = \sqrt{\frac{2E_0}{M\omega_0^2}}\cos\phi,\ \ p_0 = \sqrt{2ME_0}\sin\phi \tag{6}$$

from (5) we obtain

$$E_1 = E_0(\alpha\cos^2\phi + \beta\sin^2\phi + \gamma\sin 2\phi), \tag{7}$$

where

$$\alpha = \frac{c^2}{M^2 w_0^2} + a^2\frac{\omega_1^2}{\omega_0^2},\ \ \beta = d^2 + \omega_1^2 M^2 b^2,$$
$$\gamma = \frac{cd}{M\omega_0} + abM\frac{\omega_1^2}{\omega_0}. \tag{8}$$

Given the uniform probability distribution of initial angles ϕ equal to $1/(2\pi)$, which defines our initial ensemble at time $t = 0$, we can now calculate the averages. Thus

$$\bar{E}_1 = \frac{1}{2\pi}\oint E_1 d\phi = \frac{E_0}{2}(\alpha + \beta). \tag{9}$$

That yields $E_1 - \bar{E}_1 = E_0(\delta\cos 2\phi + \gamma\sin 2\phi)$ and

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2}\left(\delta^2 + \gamma^2\right), \tag{10}$$

where we have denoted $\delta = (\alpha - \beta)/2$.

It follows from (8), (9) that we can write (10) also in the form

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - \frac{\omega_1^2}{\omega_0^2}\right]. \tag{11}$$

It is straightforward to show that for arbitrary positive integer m, we have $\overline{(E_1 - \bar{E}_1)^{2m-1}} = 0$ and

$$\overline{(E_1 - \bar{E}_1)^{2m}} = \frac{(2m-1)!!}{m!}\left(\overline{(E_1 - \bar{E}_1)^2}\right)^m. \tag{12}$$

Thus $2m$-th moment of $P(E_1)$ is equal to $(2m - 1)!!\mu^{2m}/m!$, and therefore, indeed, all moments of $P(E_1)$ are uniquely determined by the first moment $\bar{E}_1$.

Expression (11) is positive definite by definition and this leads to the first interesting conclusion: In full generality (no restrictions on the function $\omega(t)$!) we have always $\bar{E}_1 \geq E_0\omega_1/\omega_0$ and therefore the final value of the adiabatic invariant $I_1 = \bar{E}_1/\omega_1$ is always greater or equal to the initial value $I_0 = E_0/\omega_0$. In other words, the value of the adiabatic invariant never decreases, which is a kind of irreversibility statement. Moreover, it is constant only for infinitely slow processes $T = \infty$, which is an ideal adiabatic process, i.e. $\mu = 0$. For periodic processes $\omega_1 = \omega_0$ we see that always $\bar{E}_1 \geq E_0$, so the mean energy never decreases. The other extreme to $T = \infty$ is the instantaneous ($T = 0$) jump where ω_0 switches to ω_1 discontinuously, whilst q and p remain continuous, and this results in $a = d = 1$ and $b = c = 0$, and then we find

$$\bar{E}_1 = \frac{E_0}{2}(\frac{\omega_1^2}{\omega_0^2} + 1), \qquad \mu^2 = \frac{E_0^2}{8}\left[\frac{\omega_1^2}{\omega_0^2} - 1\right]^2. \tag{13}$$

Below we shall treat the special case with $\omega_1^2 = 2\omega_0^2$, and thus will find $\mu^2/E_0^2 = 1/8 = 0.125$.

Our general study now focuses on the calculation of the transition map (4), namely its matrix elements a, b, c, d. Starting from the Hamilton function (2) and its Newton equation (1) we consider two linearly independent solutions $\psi_1(t)$ and $\psi_2(t)$ and introduce the matrix

$$\Psi(t) = \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ M\dot{\psi}_1(t) & M\dot{\psi}_2(t) \end{pmatrix}. \tag{14}$$

Consider a solution $\hat{q}(t)$ of (1) such that

$$\hat{q}(t_0) = q_0, \quad \dot{\hat{q}}(t_0) = p_0/M. \tag{15}$$

Because ψ_1 and ψ_2 are linearly independent, we can look for $\hat{q}(t)$ in the form

$$\hat{q}(t) = A\psi_1(t) + B\psi_2(t). \tag{16}$$

Then A and B are determined by

$$\begin{pmatrix} A \\ B \end{pmatrix} = \Psi^{-1}(t_0)\begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \tag{17}$$

Let $q_1 = \hat{q}(t_1)$, $\quad p_1 = M\dot{\hat{q}}(t_1)$. Then from (15)–(17) we see that

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0)\begin{pmatrix} q_0 \\ p_0 \end{pmatrix}. \tag{18}$$

We recognize the matrix on the right-hand side of (18) as the transition map Φ, that is,

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi(t_1)\Psi^{-1}(t_0). \tag{19}$$

Due to lack of space we mention only one specific model, namely the linear model defined by the piecewise linear $\omega^2(t)$ function

$$\omega^2(t) = \begin{cases} \omega_0^2 & \text{if} \quad t \leq 0 \\ \omega_0^2 + \frac{(\omega_1^2 - \omega_0^2)}{T}\, t & \text{if} \quad 0 < t < T \\ \omega_1^2 & \text{if} \quad t \geq T \end{cases}. \tag{20}$$

In this case the equation (1) can be solved exactly in terms of the Airy functions, and the formalism explained above leads in a straightforward but lengthy manner to the final exact result for $\bar{E}_1$ and consequently for μ^2 etc. It is too complex to be shown here. The special case $\omega_0^2 = 1$ and $\omega_1^2 = 2$ has been checked very carefully, also numerically, and μ^2 goes correctly from $1/8$ at $T = 0$ to zero as $T \to \infty$, in a typical oscillatory way. Using the well known asymptotic expressions for the Airy functions we find the leading asymptotic approximation

$$\frac{\mu^2}{E_0^2} = \frac{\overline{(E_1 - \bar{E}_1)^2}}{E_0^2} \approx \frac{\epsilon^2}{128} \left(9 - 4\sqrt{2}\, \cos(\frac{4 - 8\sqrt{2}}{3\,\epsilon}) \right), \tag{21}$$

where we introduce the adiabatic parameter $\epsilon = 1/T$ which is assumed small. Please observe the oscillatory approach to zero as $\epsilon \to 0$, which in the mean goes to zero quadratically as ϵ^2.

Returning to the general case we now mention that the final energy distribution function written down as

$$P(E_1) = \frac{1}{2\pi} \sum_{j=1}^{4} \left| \frac{d\phi}{dE_1} \right|_{\phi = \phi_j(E_1)} \tag{22}$$

cannot be calculated analytically in a closed form in any useful way, because it boils down to finding the roots of a quartic polynomial, so we do not try to do that here, although numerically it shows interesting aspects. It has a finite interval as its support, between the lower limit E_{min} and the upper limit E_{max}, and at both values it has an integrable singularity of the type $1/\sqrt{x}$. In between for every value of $E_1 = const = E_1(\phi)$, this equation has four solutions, namely $\phi_1, \phi_2, \phi_3, \phi_4$, and thus we have to sum up all four contributions in the general formula (22). On the other hand, as we have seen, the moments of this interesting distribution function can be calculated exactly to all orders.

We proceed with the calculation of the transition map Φ in the general case, and because (1) is generally not solvable, we have ultimately to resort to some approximations. Since the adiabatic limit $\epsilon \to 0$ is the asymptotic regime that we would like to understand, the application of the rigorous WKB theory (up to all orders) is most convenient, and usually it turns out that the leading asymptotic terms are well described by just the leading WKB terms.

We introduce re-scaled and dimensionless time λ

$$\lambda = \epsilon t, \qquad \epsilon = 1/T \tag{23}$$

so that (1) is transformed to the equation

$$\epsilon^2 q''(\lambda) + \omega^2(\lambda) q(\lambda) = 0. \tag{24}$$

Let $q_+(\lambda)$ and $q_-(\lambda)$ be two linearly independent solutions of (24). Then the matrix (14) takes the form

$$\Psi_\lambda = \begin{pmatrix} q_+(\lambda) & q_-(\lambda) \\ \epsilon M q'_+(\lambda) & \epsilon M q'_-(\lambda) \end{pmatrix} \tag{25}$$

and taking into account that $\lambda_0 = \epsilon t_0, \lambda_1 = \epsilon t_1$ we obtain for the matrix (19) the expression

$$\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi_\lambda(\lambda_1) \Psi_\lambda^{-1}(\lambda_0). \tag{26}$$

We now use the WKB method in order to obtain the coefficients a, b, c, d of the matrix Φ. To do so, we look for solution of (24) in the form

$$q(\lambda) = w \exp\left\{ \frac{1}{\epsilon} \sigma(\lambda) \right\} \tag{27}$$

where $\sigma(\lambda)$ is a complex function that satisfies the differential equation

$$(\sigma'(\lambda))^2 + \epsilon \sigma''(\lambda) = -\omega^2(\lambda) \tag{28}$$

and w is some constant with dimension of length. The WKB expansion for the phase is

$$\sigma(\lambda) = \sum_{k=0}^{\infty} \epsilon^k \sigma_k(\lambda). \tag{29}$$

Substituting (29) into (28) and comparing like powers of ϵ gives the recursion relation

$$\sigma_0'^2 = -\omega^2(\lambda), \qquad \sigma_n' = -\frac{1}{2\sigma_0'} \left(\sum_{k=1}^{n-1} \sigma_k' \sigma_{n-k}' + \sigma_{n-1}'' \right). \tag{30}$$

Here we apply our WKB notation and formalism [5] and we can choose $\sigma'_{0,+}(\lambda) = \mathrm{i}\omega(\lambda)$ or $\sigma'_{0,-}(\lambda) = -\mathrm{i}\omega(\lambda)$. That results in two linearly independent solutions of (24) given by the WKB expansions with the coefficients

$$\sigma_{0,\pm}(\lambda) = \pm \mathrm{i} \int_{\lambda_0}^{\lambda} \omega(x) dx, \;\; \sigma_{1,\pm}(\lambda) = -\frac{1}{2} \log \frac{\omega(\lambda)}{\omega(\lambda_0)},$$
$$\sigma_{2,\pm} = \pm \frac{\mathrm{i}}{8} \int_{\lambda_0}^{\lambda} \frac{3\,\omega'(x)^2 - 2\,\omega(x)\,\omega''(x)}{\omega(x)^3} dx, \;\; \ldots \tag{31}$$

Since $\omega(\lambda)$ is a real function we deduce from (30) that all functions σ'_{2k+1} are real and all functions σ'_{2k} are pure imaginary and $\sigma'_{2k,+} = -\sigma'_{2k,-}$, $\sigma'_{2k+1,+} = \sigma'_{2k+1,-}$ where $k = 0, 1, 2, \ldots$, and thus we have $\sigma'_+ = A(\lambda) + \mathrm{i}B(\lambda)$, $\sigma'_- = A(\lambda) - \mathrm{i}B(\lambda)$ where $A(\lambda) = \sum_{k=0}^{\infty} \epsilon^{2k+1} \sigma'_{2k+1}(\lambda)$, $B(\lambda) = -\mathrm{i} \sum_{k=0}^{\infty} \epsilon^{2k} \sigma'_{2k,+}(\lambda)$. Integration of the above equations yields

$$\sigma_+ = r(\lambda) + \mathrm{i}s(\lambda), \quad \sigma_- = r(\lambda) - \mathrm{i}s(\lambda), \tag{32}$$

where $r(\lambda) = \int_{\lambda_0}^{\lambda} A(x)\,dx$, $\quad s(\lambda) = \int_{\lambda_0}^{\lambda} B(x)\,dx$. Below we shall denote $s_1 = s(\lambda_1)$.

Using this notation we find that the elements of the transition matrix Φ have the following form, after taking into account that $\det(\Phi) = ab - cd = 1$,

$$\begin{aligned}
a =& -\frac{1}{\sqrt{B_0 B_1}}\left[A_0 \sin\left(\frac{s_1}{\epsilon}\right) - B_0 \cos\left(\frac{s_1}{\epsilon}\right)\right],\\
b =& \frac{1}{M\sqrt{B_0 B_1}} \sin\left(\frac{s_1}{\epsilon}\right),\\
c =& -\frac{M}{\sqrt{B_0 B_1}}\left[(A_0 A_1 + B_0 B_1)\sin\left(\frac{s_1}{\epsilon}\right)\right.\\
& \left. + (A_0 B_1 - A_1 B_0)\cos\left(\frac{s_1}{\epsilon}\right)\right],\\
d =& \frac{1}{\sqrt{B_0 B_1}}\left[A_1 \sin\left(\frac{s_1}{\epsilon}\right) + B_1 \cos\left(\frac{s_1}{\epsilon}\right)\right].
\end{aligned} \tag{33}$$

This is so far exact result, based on the WKB expansion technique. What we are mostly interested in is the asymptotic behaviour of μ^2 when ϵ is small and tends to zero. All other aspects and technical details will be published in a separate paper [4].

Let us consider the first order WKB approximation, that is,

$$A(\lambda) \approx \epsilon \sigma'_{1,+}(\lambda), \quad B(\lambda) \approx \frac{\sigma'_{0,+}(\lambda)}{\mathrm{i}} = \omega(\lambda). \tag{34}$$

We find for the variance (11)

$$\begin{aligned}
\frac{\mu^2}{E_0^2} = \epsilon^2 &\left(\frac{\omega_1^2 {\omega'_0}^2}{8\omega_0^6} + \frac{{\omega'_1}^2}{8\omega_0^2\omega_1^2} - \right.\\
&\left. \frac{\omega'_0 \omega'_1}{4\omega_0^4} \cos\left(\frac{2}{\epsilon}\int_{\lambda_0}^{\lambda_1} \omega(x)\,dx\right)\right) + O(\epsilon^3).
\end{aligned} \tag{35}$$

Substituting into the last formula $\omega(\lambda) = \sqrt{1+\lambda}$ we obtain exactly the approximation (21).

Suppose now that all derivatives at λ_0 and λ_1 vanish up to order $(n-1)$, i.e. $\omega'(\lambda_0) = \omega'(\lambda_1) = \cdots = \omega^{(n-1)}(\lambda_0) = \omega^{(n-1)}(\lambda_1) = 0$, and $\omega^{(n)}(\lambda_0)\omega^{(n)}(\lambda_1) \neq 0$. Then $\sigma'_1(\lambda_0) = \sigma'_1(\lambda_1) = \cdots = \sigma'_{n-1}(\lambda_0) = \sigma'_{n-1}(\lambda_1) = 0$, $\sigma'_n(\lambda_0)\sigma_n(\lambda_1) \neq 0$.

Hence, in the case $n = 2k-1$ we can assume

$$\begin{aligned}
A(\lambda) =& \epsilon^{2k-1}\sigma'_{2k-1,+}(\lambda) + h.o.t.\\
B(\lambda) =& \omega(\lambda) - \mathrm{i}\epsilon^{2k}\sigma'_{2k,+}(\lambda) + h.o.t.
\end{aligned} \tag{36}$$

and obtain

$$\begin{aligned}
\frac{\mu^2}{E_0^2} = \epsilon^{4k-2} &\left(\frac{\sigma'_{2k-1,+}(\lambda_1)^2}{2\omega_0^2} + \frac{\omega_1^2 \sigma'_{2k-1,+}(\lambda_0)^2}{2\omega_0^4} - \right.\\
&\left. \frac{\omega_1 \sigma'_{2k-1,+}(\lambda_0)\sigma'_{2k-1,+}(\lambda_1)}{\omega_0^3} \cos\left(\frac{2s_1}{\epsilon}\right)\right)\\
&+ O(\epsilon^{4k-1}).
\end{aligned} \tag{37}$$

In the case when $n = 2k$ we can suppose

$$\begin{aligned}
A(\lambda) =& \epsilon^{2k+1}\sigma'_{2k+1,+}(\lambda) + h.o.t.\\
B(\lambda) =& \omega(\lambda) - \mathrm{i}\epsilon^{2k}\sigma'_{2k,+}(\lambda) + h.o.t.
\end{aligned} \tag{38}$$

Then, similarly as above, we obtain

$$\begin{aligned}
\frac{\mu^2}{E_0^2} = -\epsilon^{4k} &\left(\frac{\sigma'_{2k,+}(\lambda_1)^2}{2\omega_0^2} + \frac{\omega_1^2 \sigma'_{2k,+}(\lambda_0)^2}{2\omega_0^4} - \right.\\
&\left. \frac{\omega_1 \sigma'_{2k,+}(\lambda_0)\sigma'_{2k,+}(\lambda_1)}{\omega_0^3} \cos\left(\frac{2s_1}{\epsilon}\right)\right) + O(\epsilon^{4k+1}).
\end{aligned} \tag{39}$$

From this we can conclude that if $\omega(t)$ is of class $\mathcal{C}^m$ (having m continuous derivatives, $m = n-1$) μ^2 goes to zero oscillating but in the mean as $\propto \epsilon^{2n} = \epsilon^{2(m+1)}$. If $m = \infty$ (analytic functions) according to Landau and Lifshitz [2] the decay to zero is oscillating and on the average is exponential $\propto \exp(-const/\epsilon)$.

Acknowledgments

This work was supported by the Ministry of Higher Education, Science and Technology of the Republic of Slovenia, Nova Kreditna Banka Maribor and Telekom Slovenije.

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